

# Invariant measures on multimode quantum Gaussian states

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We derive the invariant measure on the manifold of multimode quantum Gaussian states, induced by the Haar measure on the group of Gaussian unitary transformations. To this end, by introducing a bipartition of the system in two disjoint subsystems, we use a parameterization highlighting the role of nonlocal degrees of freedom – the symplectic eigenvalues – which characterize quantum entanglement across the given bipartition. A finite measure is then obtained by imposing a physically motivated energy constraint. By averaging over the local degrees of freedom we finally derive the invariant distribution of the symplectic eigenvalues in some cases of particular interest for applications in quantum optics and quantum information.

## I. INTRODUCTION

The development of quantum information theory has stimulated and motivated the study of the underlying geometry of quantum states and operations [6]. In particular, the analysis of the geometry of states describing composite quantum systems is fundamental because of the consequences of local and nonlocal transformations for the properties of correlations and entanglement [4, 5, 15]. A typical problem in quantum information is the characterization of the entanglement features across a given partition of a quantum system into two (or more) subsystems. As a matter of fact, entanglement is the most important resource for quantum computation and quantum information. Nevertheless, it is not yet completely understood. Therefore, it is of utmost importance to obtain normal forms for bipartite quantum states, in which the role of the nonlocal degrees of freedom, characterizing the entanglement, is highlighted. For the case of a quantum system described by a Hilbert space of finite dimension  $\mathcal{H} \simeq \mathbb{C}^n$ , and a bipartition  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  with  $\mathcal{H}_A \simeq \mathbb{C}^{n_A}$ ,  $\mathcal{H}_B \simeq \mathbb{C}^{n_B}$  and  $n_A + n_B = n$ , a normal form of fundamental importance is given by the Schmidt decomposition [26] of a normalized vector  $|\psi\rangle \in \mathcal{H}$ , representing a pure state of the system:

$$|\psi\rangle = \sum_{j=1}^{n_A} \sqrt{p_j} |j\rangle_A \otimes |j\rangle_B.$$

Here  $\{|j\rangle_A\}_{j=1,\dots,n_A}$ ,  $\{|j\rangle_B\}_{j=1,\dots,n_B}$  are orthonormal systems of the local Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively, and we have assumed, with no loss of generality, that  $n_A \leq n_B$ . All states can be reduced to such a form by acting with local unitary transformations on the subsystems. The components of the probability vector  $p = (p_1, \dots, p_{n_A}) \in \Delta_{n_A-1}$ , where  $\Delta_{n_A-1}$  is the  $(n_A - 1)$ -simplex ( $p_k \geq 0$ ,  $\sum p_k = 1$ ), are known as the Schmidt coefficients. They completely characterize the nonlocal features of entanglement in pure states [35].

In the study of entanglement in bipartite systems, also in view of applications [16], one is interested in the typical features of entanglement [17], where the notion of typicality is defined according to a suitable probability measure. In particular, when dealing with finite-dimensional Hilbert spaces, one is interested in the distribution of the Schmidt coefficients. Although a certain arbitrariness exists in the choice of this distribution, the natural and most unbiased distribution is the one induced by the Haar measure on the relevant unitary group. For a system with Hilbert space  $\mathcal{H} \simeq \mathbb{C}^n$ , this is the  $n$ -dimensional unitary group  $U(n)$ . The corresponding induced probability measure of the Schmidt coefficients is [19, 20]

$$d\mu_S(p) = P(p) dp = C_{n,n_A} \prod_{h>k=1}^{n_A} (p_h - p_k)^2 \prod_{j=1}^{n_A} p_j^{n_B - n_A} dp, \quad (1)$$

with  $dp$  the Lebesgue measure on  $\Delta_{n_A-1}$ , and  $C_{n,n_A}$  a normalization factor. The probability measure  $P(p)$  is by construction invariant under the action of the unitary group  $U(n)$ , that is, it is independent of the reference frame. This expression has been the starting point of several studies concerning the typicality of entanglement, see e.g. [10–12, 14, 21, 24]. We stress that the derivation of a measure for a (physically relevant) class of states is fundamental for the study and comprehension of the features of entanglement. Moreover, these kinds of studies can shed light on interesting details of the geometric structure of the Hilbert spaces used to describe quantum systems and the properties of unitary groups describing quantum dynamics.

In the case of Hilbert spaces with infinite dimensions  $\mathcal{H} = L^2(\mathbb{R})^{\otimes n} \simeq L^2(\mathbb{R}^n)$ , the limit of the invariant distribution (1) is not well defined. Then, in order to obtain a well defined distribution, and mostly for their relevance in physical applications, here we consider a specific finite-dimensional, yet unbounded, manifold  $\mathcal{G} \subset \mathcal{H}$  of pure states, known as Gaussian states [9, 32–34, 36].

Entanglement typicality in multimode Gaussian states has been the subject of previous investigations [1, 29–31], whose starting points were the probability measures induced by microcanonical-like and canonical-like ensembles and the concept of “concentration of measures”. Here we follow a different route and derive a different probability measure. More specifically, by merely relying on the symmetries of the manifold of Gaussian states, we derive the invariant measure induced by the Haar measure on the group of Gaussian unitary transformations, whose orbit passing through the vacuum state is the manifold of Gaussian states, see e.g. [33], and [22].

Furthermore, in view of physical applications, we introduce a suitable effective cutoff by means of an energy constraint that enables us to normalize the invariant measure, then to compute the average over the local degrees of freedom and finally to derive the probability distribution of the nonlocal parameters (symplectic eigenvalues) characterizing bipartite entanglement in Gaussian states. As an example, we consider a specific submanifold of Gaussian states which is most relevant for applications in the domain of quantum optics.

## II. DEFINITIONS AND MAIN RESULTS

### A. Multimode Gaussian States

In this SubSection we briefly recall few basic definitions and properties of Gaussian states. For a comprehensive review of their mathematical features and physical relevance we refer to [9, 32–34, 36].

**Definition 1** A continuous variable (CV) quantum system is defined by a collection of  $n$  bosonic modes with canonical annihilation and creation operators

$$a_k, a_k^\dagger \quad k = 1, \dots, n,$$

obeying the canonical commutation relations

$$[a_h, a_k^\dagger] = \delta_{hk}, \quad [a_h, a_k] = [a_h^\dagger, a_k^\dagger] = 0 \quad \forall h, k = 1 \dots n,$$

where  $^\dagger$  denotes the adjoint operator. The canonical operators act on the Hilbert space  $\mathcal{H} = \text{span}\{|\{j_1, j_2, \dots, j_n\}\rangle\}_{j_k=0,1,\dots,\infty}$ , where the vacuum state  $|0\rangle := |\{0, 0, \dots, 0\}\rangle$  is characterized by  $a_k|0\rangle = 0$  for all  $k = 1, \dots, n$ , and

$$|\{j_1, j_2, \dots, j_n\}\rangle = \prod_{k=1}^n (j_k!)^{-1/2} (a_k^\dagger)^{j_k} |0\rangle.$$

In the following we will study the action of multimode Gaussian unitaries, namely those unitary operators generated by quadratic self-adjoint polynomials of the canonical operators.

**Definition 2** Let us consider a Hamiltonian, quadratic in the canonical operators,

$$H_G = H_0 + H_1 + H_2, \tag{2}$$

with

$$H_0 = \theta, \quad (3)$$

$$H_1 = i \sum_{k=1}^n \left( \xi_k a_k^\dagger - \xi_k^* a_k \right), \quad (4)$$

$$H_2 = \sum_{h,k=1}^n \left( M_{hk} a_h a_k + M_{hk}^* a_h^\dagger a_k^\dagger + N_{hk} a_h^\dagger a_k \right), \quad (5)$$

where  $\theta \in \mathbb{R}$ ,  $\xi \in \mathbb{C}^n$ ,  $M$  is a complex and symmetric matrix,  $N$  is a Hermitian matrix, and  $i$  denotes the imaginary unit. We define as multimode Gaussian unitary (or  $n$ -mode Gaussian unitary) the exponential of  $H_G$ :

$$\mathcal{U}_G = e^{-iH_G}. \quad (6)$$

The set of  $n$ -mode Gaussian unitaries is a group, denoted as the *group of  $n$ -mode Gaussian unitaries*. The quadratic Hamiltonians of the form (2) constitute the Lie algebra of this group. It is well known that the group of  $n$ -mode Gaussian unitaries is a projective representation of the inhomogeneous symplectic group  $\text{ISp}(2n, \mathbb{R})$ , with the quadratic terms (5) corresponding to the homogeneous subgroup  $\text{Sp}(2n, \mathbb{R})$ , and the linear terms (4) to the subgroup of translations  $T_{2n}$ , isomorphic to  $\mathbb{C}^n$ . We hence denote as *homogeneous Gaussian unitaries* the Gaussian unitary transformations obtained by putting  $\xi = 0$ .

**Definition 3** We call any vector state of the form

$$|\psi_G\rangle = \mathcal{U}_G |0\rangle = e^{-iH_G} |0\rangle,$$

for some multimode Gaussian unitary  $\mathcal{U}_G$ , a Gaussian pure state. The set of such states will be denoted by  $\mathcal{G}$ .

The manifold of Gaussian pure states  $\mathcal{G} \subset \mathcal{H}$  is hence the orbit of the action of the group of Gaussian unitary operators on the vacuum state. The orbit of the subgroup of homogeneous Gaussian unitaries is a submanifold of Gaussian states, that we refer to as the submanifold of *homogeneous Gaussian states*. Homogeneous Gaussian states are characterized by the conditions

$$\langle \psi_G | a_k | \psi_G \rangle = \langle \psi_G | a_k^\dagger | \psi_G \rangle = 0, \quad k = 1, \dots, n.$$

According to the definition of Perelomov [25] Gaussian states are coherent states for  $\text{ISp}(2n, \mathbb{R})$ , and the homogenous Gaussian states are coherent states for  $\text{Sp}(2n, \mathbb{R})$ .

Let us hence consider a bipartition of the system into two disjoint subsets, denoted  $A$  and  $B$ , of the canonical modes,

$$\begin{aligned} a_{Ak}, a_{Ak}^\dagger, & \quad k = 1, \dots, n_A, \\ a_{Bk}, a_{Bk}^\dagger, & \quad k = 1, \dots, n_B, \end{aligned}$$

with  $n_A + n_B = n$ . This naturally induces on the total Hilbert space the bipartition into two disjoint local Hilbert spaces,  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ . Without loss of generality we assume  $n_A \leq n_B$ . For Gaussian pure states of a bipartite system the following Proposition holds:

**Proposition 1 ([7])** By acting with local Gaussian unitary transformations  $\mathcal{U}_G^A$  and  $\mathcal{U}_G^B$  on subsystems  $A$  and  $B$ , any Gaussian pure state  $|\psi_G\rangle$  of  $n_A + n_B$  modes can be put into the canonical form

$$\begin{aligned} |\psi_G^c\rangle &= \mathcal{U}_G^A \otimes \mathcal{U}_G^B |\psi_G\rangle \\ &= \prod_{k=1}^{n_A} \exp \left( r_k a_{Ak} a_{Bk} - r_k a_{Ak}^\dagger a_{Bk}^\dagger \right) |0\rangle, \end{aligned} \quad (7)$$

where  $r_1, r_2, \dots, r_{n_A}$  are real and non-negative parameters (unique up to permutations) associated to the state  $|\psi_G\rangle$ .

In order to clarify our notation, let us consider the case  $n = 2$ , with  $n_A = n_B = 1$ . Then any two mode Gaussian state, up to local Gaussian unitaries, can be put in the form

$$|\text{TMSV}\rangle = \exp\left(r a_A a_B - r a_A^\dagger a_B^\dagger\right) |0\rangle, \quad (8)$$

which is the so-called *two-mode squeezed vacuum* (or *twin-beam state*), a state of utmost importance in quantum optics and in CV implementations of quantum information processing [9].

By expanding the exponential in Eq. (8) one gets

$$|\text{TMSV}\rangle = \sqrt{\frac{2}{\nu+1}} \sum_{j=0}^{\infty} \left(\frac{\nu-1}{\nu+1}\right)^{j/2} |j\rangle \otimes |j\rangle,$$

where

$$|j\rangle = (j!)^{-1/2} (a^\dagger)^j |0\rangle,$$

and  $\nu = \cosh 2r$ . By taking the partial trace over one of the two modes, one gets a thermal-like reduced state

$$\rho_A = \rho_B = \frac{2}{\nu+1} \sum_{j=0}^{\infty} \left(\frac{\nu-1}{\nu+1}\right)^j |j\rangle \langle j|.$$

The entanglement of the two-mode squeezed vacuum is hence characterized by the single parameter  $\nu \in [1, +\infty)$ .

Coming back to the multimode case, by means of the canonical decomposition in Eq. (7), we see that it is possible to write any Gaussian pure state of  $n_A + n_B$  modes as the direct product of  $n_A$  two-mode squeezed vacua and  $n_B - n_A$  field vacua.

**Definition 4** Given a Gaussian pure state  $|\psi_G\rangle \in \mathcal{G}$ , and a bipartition into  $n_A + n_B$  modes, with  $n_A \leq n_B$ , the symplectic eigenvalues associated to the given bipartition are defined as

$$\nu_k := \cosh 2r_k, \quad k = 1, \dots, n_A, \quad (9)$$

where the parameters  $r_k$  are given by its canonical form (7).

According to Proposition 1, a bipartition of the system induces a parameterization of the manifold of Gaussian states in terms of a set of *non-local* parameters — the symplectic eigenvalues —, and of a certain set of *local* parameters. The latter correspond to the local unitaries  $\mathcal{U}_G^A, \mathcal{U}_G^B$  acting on the subsystems, and will be denoted as  $\alpha_A$  and  $\alpha_B$ .

All the entanglement properties of a  $n$ -mode Gaussian pure state with respect to the given bipartition are uniquely determined by the  $n_A$  symplectic eigenvalues. The latter are invariant under local unitary transformations and can be explicitly computed by symplectic diagonalization [7, 18].

## B. Main results

The manifold  $\mathcal{G}$  of Gaussian states is an orbit of the group of Gaussian unitary transformations. Therefore, there exists a unique invariant measure on the manifold of Gaussian states induced by the Haar measure on the group of Gaussian unitaries. Our main interest is in computing the probability measure of the symplectic eigenvalues,  $\nu_k \in [1, +\infty)$ , induced by that invariant measure.

The following Theorem is the main result of this paper:

**Theorem 1** The invariant measure on the manifold of Gaussian pure states has the form

$$d\mu_G = K_{n,n_A} \prod_{h>k=1}^{n_A} (\nu_h^2 - \nu_k^2)^2 \prod_{j=1}^{n_A} \nu_j^2 (\nu_j^2 - 1)^{n_B - n_A} d\nu d\mu_A(\alpha_A) d\mu_B(\alpha_B) d\theta, \quad (10)$$

where:  $\nu = (\nu_1, \dots, \nu_{n_A})$  are the symplectic eigenvalues and  $d\nu = \prod_{h=1}^{n_A} d\nu_h$ ;  $d\mu_A(\alpha_A)$ ,  $d\mu_B(\alpha_B)$  denote the factors in the invariant measure depending on the local degrees of freedom  $\alpha_A$ ,  $\alpha_B$ , corresponding to the local Gaussian unitaries acting respectively on subsystems  $A$ ,  $B$ ;  $d\theta$  corresponds to the scalar term in Definition 2; and  $K_{n,n_A}$  is a normalization factor.

The proof of Theorem 1 is given in Section IV A.

This expression for the invariant measure on the manifold of  $n$ -mode Gaussian states makes use of a parameterization which highlights the symplectic eigenvalues as signatures of the entanglement across a given bipartition of the system.

We also compute an explicit expression for the Haar measure on the group of Gaussian unitaries. For the sake of conciseness and in view of the physical applications we restrict to the subgroup of homogenous Gaussian unitary transformations, obtained by setting  $\xi = 0$  in Eq. (2). The generated Gaussian unitary can be written according to the Euler decomposition:

$$\mathcal{U}_G = e^{-i\theta} \exp \left( -i \sum_{i,j=1}^n T_{ij} a_i^\dagger a_j \right) \exp \left( \sum_{k=1}^n s_k a_k^2 - s_k (a_k^\dagger)^2 \right) \exp \left( -i \sum_{i,j=1}^n T'_{ij} a_i^\dagger a_j \right), \quad (11)$$

where  $\theta \in \mathbb{R}$ ,  $T$  and  $T'$  are Hermitian matrices, and  $s_1, s_2, \dots, s_n$  are real and non-negative parameters (unique up to permutations). Let us recall that the Gaussian unitaries of the form  $\mathcal{U}_{\bar{G}} = \exp \left( -i \sum_{i,j=1}^n T_{ij} a_i^\dagger a_j \right)$ , with  $T$  Hermitian, define a representation of the group  $U(n)$ , see e.g. [3]. In particular, the action of the Gaussian unitary on the canonical operators reads

$$\mathcal{U}_{\bar{G}}^\dagger a_k \mathcal{U}_{\bar{G}} = \sum_{h=1}^n U_{kh} a_h, \quad (12)$$

where  $U = e^{-iT} \in U(n)$ . Furthermore,

$$\exp \left( - \sum_{j=1}^n s_j a_j^2 - s_j (a_j^\dagger)^2 \right) a_k \exp \left( \sum_{j=1}^n s_j a_j^2 - s_j (a_j^\dagger)^2 \right) = \cosh(2s_k) a_k - \sinh(2s_k) a_k^\dagger. \quad (13)$$

Then the following Theorem holds:

**Theorem 2** *The Haar invariant measure on the group of  $n$ -mode homogeneous Gaussian unitaries defined in Eq. (11) is given by*

$$d\mu(\mathcal{U}_G) = K_n \prod_{h < k=1}^n |\lambda_h - \lambda_k| d\lambda d\mu(U) d\mu(U'),$$

where  $\mu$  denotes the Haar invariant measure on the unitary group  $U(n)$ ,  $U = \exp(-iT)$ ,  $U' = \exp(-iT')$ ,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\lambda_k = \cosh 2s_k$ ,  $d\lambda = \prod_{k=1}^n d\lambda_k$ , and  $K_n$  is a normalization factor.

The proof of Theorem 2 is given in Section IV B.

### III. APPLICATIONS AND EXAMPLES

#### A. A finite measure from an energy constraint

The manifold of Gaussian states is unbounded, implying that the invariant measure cannot be normalized globally. As a consequence, all the statistical moments of the symplectic eigenvalues diverge. This also yields unbounded statistical moments of entropies of entanglement [2]. The same holds true for other entanglement quantifiers, as for instance the logarithmic negativity [27] and the coherent information [23], which are increasing functions of the symplectic eigenvalues. In order to avoid unphysical results we impose a suitable constraint, yielding an effective cutoff on the unbounded manifold of Gaussian states, see also [13, 18]. Here we choose to constrain the mean value of the energy in each subsystem.

**Definition 5** We define the mean energies of subsystems  $A$  and  $B$  for a Gaussian pure state  $|\psi_G\rangle$ :

$$\begin{aligned}\mathcal{E}_A &= \frac{1}{2} \sum_{k=1}^{n_A} \langle \psi_G | \left( a_{A_k}^\dagger a_{A_k} + a_{A_k} a_{A_k}^\dagger \right) | \psi_G \rangle, \\ \mathcal{E}_B &= \frac{1}{2} \sum_{k=1}^{n_B} \langle \psi_G | \left( a_{B_k}^\dagger a_{B_k} + a_{B_k} a_{B_k}^\dagger \right) | \psi_G \rangle.\end{aligned}$$

For the sake of simplicity and motivated by the physical interpretation in the following we restrict to the submanifold of homogeneous Gaussian states and to the case  $n_A = n_B = n/2$  ( $n$  even). The subsystems mean energies are the quantities that we will consider fixed (and finite) on a physical basis. The mean energies take values in the interval  $[1/2, +\infty)$ . They can be given a suitable expression by virtue of the following Lemma:

**Lemma 1** The mean energies of subsystems  $A$  and  $B$ , when calculated on homogeneous Gaussian states and for  $n_A = n_B = n/2$ , have the form

$$\mathcal{E}_A = \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{A_{hk}}|^2 \lambda_{A_h} \nu_k, \quad (14)$$

$$\mathcal{E}_B = \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{B_{hk}}|^2 \lambda_{B_h} \nu_k, \quad (15)$$

where  $U_A$  and  $U_B$  are unitary matrices of  $U(n/2)$ ,  $\nu$  are the symplectic eigenvalues (9), and  $\lambda_A = (\lambda_{A1}, \lambda_{A2}, \dots, \lambda_{An/2})$ ,  $\lambda_B = (\lambda_{B1}, \lambda_{B2}, \dots, \lambda_{Bn/2})$  are the parameters defined in Theorem 2.

The proof of the Lemma is given in Section IV C. Notice that the mean energies are functions of both the nonlocal parameters  $\nu$  and the local ones,  $U_A, \lambda_A, U_B, \lambda_B$ .

By fixing the values of the local mean energies, we get an expression for the probability measure of the symplectic eigenvalues:

$$\begin{aligned}d\mu(\nu|\mathcal{E}_A, \mathcal{E}_B) &= P(\nu|\mathcal{E}_A, \mathcal{E}_B) d\nu \\ &= \int \delta\left(\mathcal{E}_A - \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{A_{hk}}|^2 \lambda_{A_h} \nu_k\right) \delta\left(\mathcal{E}_B - \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{B_{hk}}|^2 \lambda_{B_h} \nu_k\right) d\mu_G,\end{aligned}$$

where  $\delta$  denotes the Dirac function.

This integral can be evaluated by using the invariant measure of Eq. (10) after making the formal substitutions of  $d\mu(\alpha_A)$ ,  $d\mu(\alpha_B)$  with  $d\mu(\mathcal{U}_G^A)$ ,  $d\mu(\mathcal{U}_G^B)$ , the latter being the Haar invariant measures on the local groups of Gaussian unitaries. The explicit form of the Haar measures  $d\mu(\mathcal{U}_G^A)$ ,  $d\mu(\mathcal{U}_G^B)$  is given by Theorem 2. After integrating over all the cyclic variables, we finally obtain

$$P(\nu|\mathcal{E}_A, \mathcal{E}_B) = K_{n,n/2} g(\nu, \mathcal{E}_A) g(\nu, \mathcal{E}_B) \prod_{j < k=1}^{n/2} (\nu_j^2 - \nu_k^2)^2 \prod_{l=1}^{n/2} \nu_l^2, \quad (16)$$

where

$$g(\nu, \mathcal{E}_A) := K_{n/2} \int \delta\left(\mathcal{E}_A - \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{A_{hk}}|^2 \lambda_{A_h} \nu_k\right) \prod_{h < k=1}^{n/2} |\lambda_{A_h} - \lambda_{A_k}| d\lambda_A d\mu(U_A), \quad (17)$$

and  $g(\nu, \mathcal{E}_B)$  is given by the analogous expression.

Let us apply Eq. (16) to some simple but enlightening cases.

### B. The case of 1 + 1 mode

For  $n = 2$  and  $n_A = n_B = 1$ , the mean energies (14)-(15) read

$$\mathcal{E}_A = \frac{1}{2}\lambda_A\nu, \quad \mathcal{E}_B = \frac{1}{2}\lambda_B\nu.$$

Thus, the integral in (17) becomes

$$g(\nu, \mathcal{E}_A) = K_1 \int \delta\left(\mathcal{E}_A - \frac{1}{2}\lambda_{A1}\nu\right) d\lambda_{A1} = \frac{2K_1}{\nu}.$$

Therefore, after fixing the values of the normalization constants  $K_1, K_{2,1}$  the probability measure (16) of the symplectic eigenvalues reads

$$P(\nu|\mathcal{E}_A, \mathcal{E}_B) = \frac{1}{\min\{\mathcal{E}_A, \mathcal{E}_B\} - 1}, \quad \nu \in [1, \min\{\mathcal{E}_A, \mathcal{E}_B\}].$$

### C. The case of 2 + 2 modes

For  $n = 4$  and  $n_A = n_B = 2$ , the probability measure of the symplectic eigenvalues reads

$$P(\nu_1, \nu_2|\mathcal{E}_A, \mathcal{E}_B) = K_{4,2} g(\nu_1, \nu_2, \mathcal{E}_A) g(\nu_1, \nu_2, \mathcal{E}_B) (\nu_1^2 - \nu_2^2)^2 \nu_1^2 \nu_2^2,$$

with  $g$  given by (17). The  $2 \times 2$  unitary matrix acting on the subsystem  $A$  can be parameterized as

$$U_A = \begin{pmatrix} e^{i\varphi} \cos(\vartheta/2) & e^{i\chi} \sin(\vartheta/2) \\ -e^{-i\chi} \sin(\vartheta/2) & e^{-i\varphi} \cos(\vartheta/2) \end{pmatrix},$$

with  $\vartheta \in [0, \pi]$ ,  $\varphi, \chi \in [0, 2\pi]$ , and the normalized Haar measure on  $U(2)$  reads  $d\mu(U_A) = 2^{-1}(2\pi)^{-2} \sin \vartheta d\vartheta d\varphi d\chi$ . Then, from (14), (15) it follows

$$\mathcal{E}_A = \frac{1}{4}(\lambda_{A1} + \lambda_{A2})(\nu_1 + \nu_2) + \frac{1}{4}\cos \vartheta (\lambda_{A1} - \lambda_{A2})(\nu_1 - \nu_2).$$

After integration over  $d\varphi d\chi$  we get

$$\begin{aligned} g(\nu_1, \nu_2, \mathcal{E}_A) &= \frac{K_2}{2} \int d\lambda_{A1} d\lambda_{A2} d\cos \vartheta |\lambda_{A1} - \lambda_{A2}| \\ &\quad \times \delta\left(\mathcal{E}_A - \frac{1}{4}(\lambda_{A1} + \lambda_{A2})(\nu_1 + \nu_2) - \frac{1}{4}\cos \vartheta (\lambda_{A1} - \lambda_{A2})(\nu_1 - \nu_2)\right) \\ &= \frac{2K_2}{|\nu_1 - \nu_2|} \int_D d\lambda_{A1} d\lambda_{A2}. \end{aligned}$$

The domain  $D$  for the  $\lambda_A$  variables is determined by the inequality

$$\left| \frac{4\mathcal{E}_A - (\lambda_{A1} + \lambda_{A2})(\nu_1 + \nu_2)}{(\lambda_{A1} - \lambda_{A2})(\nu_1 - \nu_2)} \right| \leq 1,$$

yielding

$$\int_D d\lambda_{A1} d\lambda_{A2} = \frac{|\nu_1 - \nu_2| [2\mathcal{E}_A - (\nu_1 + \nu_2)]^2}{\nu_1 \nu_2 (\nu_1 + \nu_2)}.$$

Eventually, we obtain

$$P(\nu_1, \nu_2|\mathcal{E}_A, \mathcal{E}_B) = 4 K_{4,2} K_2^2 (\nu_1 - \nu_2)^2 [2\mathcal{E}_A - (\nu_1 + \nu_2)]^2 [2\mathcal{E}_B - (\nu_1 + \nu_2)]^2,$$

where the range of the symplectic eigenvalues is determined by the inequalities

$$\begin{cases} \nu_1 \geq 1, \\ \nu_2 \geq 1, \\ \nu_1 + \nu_2 \leq 2 \min\{\mathcal{E}_A, \mathcal{E}_B\}. \end{cases}$$

The probability measure is zero for  $\nu_1 = \nu_2$ , and attains its maximum when one of the two symplectic eigenvalues is equal to one.



### D. Nonlinear optical parametric processes

The measure (10) has been obtained by requiring the invariance over the whole manifold of Gaussian states  $\mathcal{G}$ . However, in physical applications, one is mostly interested in submanifolds of states arising from specific physical processes. The most relevant ones in which multimode Gaussian states are produced are nonlinear optical parametric processes, see e.g. [8]. In some of these processes there is a natural bipartition of the bosonic system into two disjoint subsets of canonical modes, commonly referred to as the *signal* and *idler* modes. Let  $\bar{\mathcal{G}}$  (with  $\bar{\mathcal{G}} \subset \mathcal{G}$ ) be the submanifold of such Gaussian states. The quadratic Hamiltonians generating them are of the form

$$H_{\bar{\mathcal{G}}} = \theta + \sum_{h,k=1}^{n_A} N_{Ahk} a_{Ah}^\dagger a_{Ak} + \sum_{h,k=1}^{n_B} N_{Bhk} a_{Bh}^\dagger a_{Bk} + \sum_{h=1}^{n_A} \sum_{k=1}^{n_B} \left( M_{hk} a_{Ah} a_{Bk} + M_{hk}^* a_{Ah}^\dagger a_{Bk}^\dagger \right). \quad (18)$$

where  $N_A$  and  $N_B$  are Hermitian matrices. Comparing with Eq. (2), the linear terms and some of the quadratic terms are dropped. The subgroup generated by these Hamiltonians is a projective representation of  $\text{SU}(n_A, n_B)$  [28]. According to [25], the Gaussian states of the form  $|\psi_{\bar{\mathcal{G}}}\rangle = \exp(-iH_{\bar{\mathcal{G}}})|0\rangle$  are hence coherent states for  $\text{SU}(n_A, n_B)$ . Following [28], these states can be written as

$$|\psi_{\bar{\mathcal{G}}}\rangle = e^{-i\theta} \exp \left( \sum_{h=1}^{n_A} \sum_{k=1}^{n_B} F_{hk} a_{Ah} a_{Bk} - \sum_{h=1}^{n_A} \sum_{k=1}^{n_B} F_{hk}^* a_{Ah}^\dagger a_{Bk}^\dagger \right) |0\rangle.$$

The canonical form for this class of states can be obtained from the singular value decomposition of the matrix  $F$ ,  $F_{hk} = \sum_j U_{hj} r_j V_{kj}$  with  $U \in \text{U}(n_A)$ ,  $V \in \text{U}(n_B)$  and  $r_j \geq 0$  are its singular values. By virtue of Eq. (12), we can find a pair of local Gaussian unitaries  $\mathcal{U}_{\bar{\mathcal{G}}}^A, \mathcal{U}_{\bar{\mathcal{G}}}^B$  such that  $\mathcal{U}_{\bar{\mathcal{G}}}^A a_{Ah} \mathcal{U}_{\bar{\mathcal{G}}}^A = \sum_{j=1}^{n_A} U_{hj}^* a_{Aj}$  and  $\mathcal{U}_{\bar{\mathcal{G}}}^B a_{Bk} \mathcal{U}_{\bar{\mathcal{G}}}^B = \sum_{i=1}^{n_B} V_{ki}^* a_{Bi}$ . Thus the canonical forms of these states read

$$|\psi_{\bar{\mathcal{G}}}^c\rangle = \mathcal{U}_{\bar{\mathcal{G}}}^A \otimes \mathcal{U}_{\bar{\mathcal{G}}}^B |\psi_{\bar{\mathcal{G}}}\rangle = \prod_{k=1}^{n_A} \exp \left( r_k a_{Ak} a_{Bk} - r_k a_{Ak}^\dagger a_{Bk}^\dagger \right) |0\rangle,$$

where the local Gaussian unitaries are of the form

$$\mathcal{U}_{\bar{\mathcal{G}}}^A = \exp \left( -i \sum_{h,k=1}^{n_A} T_{Ahk} a_{Ah}^\dagger a_{Ak} \right), \quad (19)$$

$$\mathcal{U}_{\bar{\mathcal{G}}}^B = \exp \left( -i \sum_{h,k=1}^{n_B} T_{Bhk} a_{Bh}^\dagger a_{Bk} \right), \quad (20)$$

with  $T_A, T_B$  Hermitian matrices.

The following Theorem holds:

**Theorem 3** *The invariant measure on  $\bar{\mathcal{G}}$  has the following expression*

$$d\mu_{\bar{\mathcal{G}}} = \bar{K}_{n,n_A} \prod_{h < k=1}^{n_A} (\nu_h - \nu_k)^2 \prod_{j=1}^{n_A} (\nu_j - 1)^{n_B - n_A} d\nu d\mu(\bar{\alpha}_A) d\mu(\bar{\alpha}_B) d\theta, \quad (21)$$

where  $\bar{K}_{n,n_A}$  is a normalization constant and the local degrees of freedom  $\bar{\alpha}_A, \bar{\alpha}_B$ , are induced by the local unitary transformations  $\mathcal{U}_{\bar{\mathcal{G}}}^A, \mathcal{U}_{\bar{\mathcal{G}}}^B$ .

The proof is given in SubSection IV D.

The expression for the mean energy of the subsystems takes a particular simple form on this submanifold, see Section IV C. For a balanced bipartition,  $n_A = n_B = n/2$  ( $n$  even),

$$\mathcal{E}_A = \mathcal{E}_B = \frac{1}{2} \sum_{k=1}^{n/2} \nu_k,$$



and the corresponding probability measure of the symplectic eigenvalues at fixed mean energy is

$$d\mu(\nu|\mathcal{E}) = \bar{K}_{n,n/2} \delta \left( \mathcal{E} - \frac{1}{2} \sum_{j=1}^{n/2} \nu_j \right) \prod_{h < k=1}^{n/2} (\nu_h - \nu_k)^2 d\nu,$$

with  $\mathcal{E} = \mathcal{E}_A = \mathcal{E}_B$ .

#### IV. PROOFS OF THE THEOREMS

##### A. Proof of Theorem 1

In order to derive the invariant measure (10) we look at the set of Gaussian pure states as a real manifold. Such a manifold is embedded in the  $n$ -mode Hilbert space, which in turn is considered as a real vector space endowed with the real scalar product induced by the standard Hermitian scalar product. We then consider a system  $\{H_\alpha\}_\alpha$  of generators of the algebra of the group of Gaussian unitary transformations, and evaluate the corresponding infinitesimal transformations on the canonical state:

$$\begin{aligned} e^{-i d\alpha H_\alpha} |\psi_G^c\rangle - |\psi_G^c\rangle &\approx -i d\alpha H_\alpha |\psi_G^c\rangle \\ &= -i d\alpha U (U_r^\dagger H_\alpha U_r) |0\rangle \\ &=: -i d\alpha U_r |\psi_{G\alpha}^c\rangle \end{aligned} \tag{22}$$

where

$$U_r = \prod_{k=1}^{n_A} \exp \left( r_k a_{Ak} a_{Bk} - r_k a_{Ak}^\dagger a_{Bk}^\dagger \right),$$

$$|\psi_{G\alpha}^c\rangle = (U_r^\dagger H_\alpha U_r) |0\rangle,$$

and  $r = (r_1, r_2, \dots, r_{n_A}) \in \mathbb{R}^{n_A}$ . Therefore, we can define a set of vector-valued one-forms  $\{d\alpha |\psi_{G\alpha}^c\rangle\}_\alpha$ , where we have neglected the unitary factor  $-iU_r$ . By construction, the volume form generated by a maximal subset of linearly independent one-forms is the desired invariant measure on the manifold of Gaussian states. To compute the latter we then proceed as follows. By looking at the Hilbert space as a real vector space, we fix a suitable system  $\{\zeta_\beta\}_\beta$ , orthonormal with respect to the real scalar product. We hence expand the one-forms in terms of these vectors, i.e.,

$$|\psi_{G\alpha}^c\rangle = \sum_{\beta} J_{\alpha\beta} \zeta_\beta.$$

Finally, the invariant measure is given in terms of the determinant of the matrix of coefficients:

$$d\mu_G = |\det J| \wedge_\alpha d\alpha.$$

In order to obtain an expression in the form of the product of a measure on the symplectic eigenvalues and a measure on the local degrees of freedom, we consider a basis for the algebra of the group of Gaussian unitary transformations composed by:

- the generator of the scalar-phase shift;
- a basis for the generators which are linear in the canonical operators;
- a basis for the quadratic generators acting on subsystem  $A$ ;
- a basis for the quadratic generators acting on subsystem  $B$ ;

- a suitable set of linearly independent nonlocal generators which change the values of the symplectic eigenvalues.

Before going into the details of the calculations, let us have a closer look at the action of the unitary transformation  $U_r$  on the canonical operators. The following identities hold:

- for  $k \leq n_A$

$$U_r^\dagger a_{A_k} U_r = \cosh r_k a_{A_k} + \sinh r_k a_{B_k}^\dagger, \quad (23)$$

$$U_r^\dagger a_{B_k} U_r = \cosh r_k a_{B_k} + \sinh r_k a_{A_k}^\dagger, \quad (24)$$

- for  $k > n_A$

$$U_r^\dagger a_{B_k} U_r = a_{B_k}. \quad (25)$$

The Lie algebra of the group of Gaussian unitaries is composed of Hamiltonians which are (at most quadratic) functions of the canonical operators,

$$H_\alpha = H_\alpha(a, a^\dagger),$$

where we have introduced the short-hand notation

$$\begin{aligned} a &= (a_{A_1}, \dots, a_{A_{n_A}}, a_{B_1}, \dots, a_{B_{n_B}}), \\ a^\dagger &= (a_{A_1}^\dagger, \dots, a_{A_{n_A}}^\dagger, a_{B_1}^\dagger, \dots, a_{B_{n_B}}^\dagger). \end{aligned}$$

Below we compute the corresponding vector-valued one-forms using the following relation

$$d\alpha|\psi_{G_\alpha}^c\rangle = d\alpha(U_r^\dagger H_\alpha(a, a^\dagger) U_r)|0\rangle = d\alpha H_\alpha(a', a'^\dagger)|0\rangle,$$

where  $a' = U_r^\dagger a U_r$ ,  $a'^\dagger = U_r^\dagger a^\dagger U_r$  are explicitly given by Eqs. (23)-(25). It follows that the vector-valued one-forms belong to the linear span of the following vectors:

- for  $k \leq n_A$

$$\begin{aligned} |k, 0, 0\rangle &:= a_{A_k}^\dagger |0\rangle, \\ |0, k, 0\rangle &:= a_{B_k}^\dagger |0\rangle, \end{aligned}$$

- for  $k > n_A$

$$|0, 0, k\rangle := a_{B_k}^\dagger |0\rangle,$$

- for  $h \leq n_A, k \leq n_A$

$$\begin{aligned} |hk, 0, 0\rangle &:= a_{A_h}^\dagger a_{A_k}^\dagger |0\rangle, \\ |0, hk, 0\rangle &:= a_{B_h}^\dagger a_{B_k}^\dagger |0\rangle, \\ |h, k, 0\rangle &:= a_{A_h}^\dagger a_{B_k}^\dagger |0\rangle, \end{aligned}$$

- for  $k \leq n_A$

$$\begin{aligned} |k_2, 0, 0\rangle &:= 2^{-1/2} (a_{A_k}^\dagger)^2 |0\rangle, \\ |0, k_2, 0\rangle &:= 2^{-1/2} (a_{B_k}^\dagger)^2 |0\rangle, \end{aligned}$$

- $h \leq n_A, k > n_A$

$$\begin{aligned} |h, 0, k\rangle &:= a_{A_h}^\dagger a_{B_k}^\dagger |0\rangle, \\ |0, h, k\rangle &:= a_{B_h}^\dagger a_{B_k}^\dagger |0\rangle. \end{aligned}$$

In the following we compute the explicit expressions of the one-forms.

### 1. The scalar term

By variation of the parameter  $\theta$ , corresponding to a scalar phase-factor, we obtain the one-form

$$d\theta|0\rangle. \quad (26)$$

### 2. The linear terms

Let us now consider the linear generators, proportional to the complex vector  $\xi$ . We consider variations of the real parameters

$$\begin{aligned} &\{\text{Re}(\xi_{A_k}), \text{Im}(\xi_{A_k})\}_{k \leq n_A}, \\ &\{\text{Re}(\xi_{B_k}), \text{Im}(\xi_{B_k})\}_{k \leq n_A}, \\ &\{\text{Re}(\xi_{B_k}), \text{Im}(\xi_{B_k})\}_{n_A < k \leq n_B}, \end{aligned}$$

where Re and Im denote the real and the imaginary part. Thus, using the identities (23)-(25), we get the following expressions for the corresponding one-forms:

- for  $k \leq n_A$

$$d\text{Re}(\xi_{A_k}) (\cosh r_k |k, 0, 0\rangle + \sinh r_k |0, k, 0\rangle), \quad (27)$$

$$\text{idIm}(\xi_{A_k}) (\cosh r_k |k, 0, 0\rangle + \sinh r_k |0, k, 0\rangle), \quad (28)$$

$$d\text{Re}(\xi_{B_k}) (\sinh r_k |k, 0, 0\rangle + \cosh r_k |0, k, 0\rangle), \quad (29)$$

$$\text{idIm}(\xi_{B_k}) (\sinh r_k |k, 0, 0\rangle + \cosh r_k |0, k, 0\rangle), \quad (30)$$

- for  $k > n_A$

$$d\text{Re}(\xi_{B_k}) |0, 0, k\rangle, \quad (31)$$

$$\text{idIm}(\xi_{B_k}) |0, 0, k\rangle. \quad (32)$$

### 3. The quadratic terms: subsystem A

We now consider the generators which are quadratic in the canonical operators of subsystem A:

$$H_A = \sum_{h,k=1}^{n_A} M_{A_{hk}} a_{A_h} a_{A_k} + M_{A_{hk}}^* a_{A_h}^\dagger a_{A_k}^\dagger + N_{A_{hk}} a_{A_h}^\dagger a_{A_k},$$

where  $M_A$  is a complex-valued symmetric matrix, and  $N_A$  is a Hermitian matrix.

Considering the variations of the parameters  $\text{Re}(M_{A_{hk}})$ ,  $\text{Im}(M_{A_{hk}})$ , for  $h \leq k \leq n_A$ , we have the one-forms

$$d\text{Re}(M_{A_{hk}}) (\cosh r_h \cosh r_k |hk, 0, 0\rangle + \sinh r_h \sinh r_k |0, hk, 0\rangle), \quad (33)$$

$$d\text{Re}(M_{A_{kk}}) \sqrt{2} (\cosh^2 r_k |k_2, 0, 0\rangle + \sinh^2 r_k |0, k_2, 0\rangle), \quad (34)$$

$$-\text{idIm}(M_{A_{hk}}) (\cosh r_h \cosh r_k |hk, 0, 0\rangle - \sinh r_h \sinh r_k |0, hk, 0\rangle), \quad (35)$$

$$-\text{idIm}(M_{A_{kk}}) \sqrt{2} (\cosh^2 r_k |k_2, 0, 0\rangle - \sinh^2 r_k |0, k_2, 0\rangle). \quad (36)$$

The variations of the parameters  $\text{Re}(N_{A_{hk}})$ ,  $\text{Im}(N_{A_{hk}})$  for  $h < k \leq n_A$ , and  $N_{A_{kk}}$  for  $k \leq n_A$ , yield

$$d\text{Re}(N_{A_{hk}}) (\cosh r_h \sinh r_k |h, k, 0\rangle + \sinh r_h \cosh r_k |k, h, 0\rangle), \quad (37)$$

$$dN_{A_{kk}} (\cosh r_k \sinh r_k |k, k, 0\rangle + \sinh^2 r_k |0\rangle), \quad (38)$$

$$\text{idIm}(N_{A_{hk}}) (\cosh r_h \sinh r_k |h, k, 0\rangle - \sinh r_h \cosh r_k |k, h, 0\rangle). \quad (39)$$

#### 4. The quadratic terms: subsystem $B$

Moving to the local transformations on the subsystem  $B$ , we first consider the quadratic Hamiltonians involving the operators  $\{a_{Bk}, a_{Bk}^\dagger\}_{k \leq n_A}$ , which are of the form

$$H_B = \sum_{h,k=1}^{n_A} M_{Bhk} a_{Bh} a_{Bk} + M_{Bhk}^* a_{Bh}^\dagger a_{Bk}^\dagger + N_{Bhk} a_{Bh}^\dagger a_{Bk} .$$

Proceeding as in the case of subsystem  $A$ , we obtain the following one-forms from the variations of the matrix  $M_B$ :

$$\text{dRe}(M_{Bhk}) (\sinh r_h \sinh r_k |hk, 0, 0\rangle + \cosh r_h \cosh r_k |0, hk, 0\rangle) , \quad (40)$$

$$\text{dRe}(M_{Bkk}) \sqrt{2} (\sinh^2 r_k |k_2, 0, 0\rangle + \cosh^2 r_k |0, k_2, 0\rangle) , \quad (41)$$

$$\text{idIm}(M_{Bhk}) (\sinh r_h \sinh r_k |hk, 0, 0\rangle - \cosh r_h \cosh r_k |0, hk, 0\rangle) , \quad (42)$$

$$\text{idIm}(M_{Bkk}) \sqrt{2} (\sinh^2 r_k |k_2, 0, 0\rangle - \cosh^2 r_k |0, k_2, 0\rangle) . \quad (43)$$

From the variations of the matrix  $N_B$  we obtain the one-forms

$$\text{dRe}(N_{Bhk}) (\sinh r_h \cosh r_k |h, k, 0\rangle + \cosh r_h \sinh r_k |k, h, 0\rangle) , \quad (44)$$

$$\text{d}N_{Bkk} (\sinh r_k \cosh r_k |k, k, 0\rangle + \sinh^2 r_k |0\rangle) , \quad (45)$$

$$-\text{idIm}(N_{Bhk}) (\sinh r_h \cosh r_k |h, k, 0\rangle - \cosh r_h \sinh r_k |k, h, 0\rangle) . \quad (46)$$

Then we consider the Hamiltonians coupling the operators  $\{a_{Bk}, a_{Bk}^\dagger\}_{k \leq n_A}$  with  $\{a_{Bk}, a_{Bk}^\dagger\}_{k > n_A}$ , that is,

$$H_B = \sum_{h=1}^{n_A} \sum_{k=n_A+1}^{n_B} \left( P_{B' hk} a_{Bh} a_{Bk} + P_{B' hk}^* a_{Bh}^\dagger a_{Bk}^\dagger + Q_{B' hk} a_{Bh} a_{Bk}^\dagger + Q_{B' hk}^* a_{Bh}^\dagger a_{Bk} \right) ,$$

where  $P_{B'}$  and  $Q_{B'}$  are complex-valued matrices. For  $h \leq n_A$  and  $n_A < k \leq n_B$  we get

$$\text{dRe}(P_{B' hk}) \cosh r_h |0, h, k\rangle , \quad (47)$$

$$\text{idIm}(P_{B' hk}) \cosh r_h |0, h, k\rangle , \quad (48)$$

$$\text{dRe}(Q_{B' hk}) \sinh r_h |h, 0, k\rangle , \quad (49)$$

$$\text{idIm}(Q_{B' hk}) \sinh r_h |h, 0, k\rangle . \quad (50)$$

Finally, let us consider the quadratic Hamiltonians containing only the operators  $\{a_{Bk}, a_{Bk}^\dagger\}_{k > n_A}$ :

$$H_{B'} = \sum_{h,k=n_A+1}^{n_B} M_{B' hk} a_{Bh} a_{Bk} + M_{B' hk}^* a_{Bh}^\dagger a_{Bk}^\dagger + N_{B' hk} a_{Bh}^\dagger a_{Bk} ,$$

where the matrix  $M_{B'}$  is complex-valued and symmetric and the matrix  $N_{B'}$  is Hermitian. We obtain that the variations of the elements of the matrix  $N_{B'}$  yield vanishing one-forms. The non-zero one-forms are generated by the variations of  $M_{B'}$ ,

$$\text{dRe}(M_{B' hk}) 2|0, 0, hk\rangle , \quad (51)$$

$$\text{dRe}(M_{B' kk}) \sqrt{2}|0, 0, k_2\rangle , \quad (52)$$

$$-\text{idIm}(M_{B' hk}) 2|0, 0, hk\rangle , \quad (53)$$

$$-\text{idIm}(M_{B' kk}) \sqrt{2}|0, 0, k_2\rangle . \quad (54)$$

### 5. The nonlocal generators

It remains to consider the nonlocal transformations whose action changes the symplectic eigenvalues. These are generated by Hamiltonians of the form

$$(H_{NL})_k = i \left( a_{Ak} a_{Bk} - a_{Ak}^\dagger a_{Bk}^\dagger \right), \quad k \leq n_A,$$

The corresponding one-forms are

$$-i dr_k |k, k, 0\rangle = \frac{-i d\nu_k}{2 \sinh 2r_k} |k, k, 0\rangle, \quad k \leq n_A. \quad (55)$$

### 6. The invariant measure

We can now compute the invariant measure on the manifold of Gaussian states.

First, we consider the one-forms corresponding to linear Hamiltonians. The matrix of coefficients is readily obtained from Eqs. (27)-(32), from which we obtain the following factor in the invariant measure

$$\prod_{h=1}^{n_A} d\text{Re}(\xi_{Ah}) d\text{Re}(\xi_{Ah}) \prod_{k=1}^{n_B} d\text{Re}(\xi_{Bk}) d\text{Re}(\xi_{Bk}) = d\xi_A d\xi_B,$$

Second, we consider the one-forms corresponding to the scalar phase-shift and the quadratic Hamiltonians. The matrix of coefficients can be straightforwardly obtained from Eqs. (26), (33)-(39), (40)-(46), (47)-(50), (51)-(54), and (55). A maximal subset of linearly independent one-forms can be obtained by eliminating the one-form in (38), which is proportional to the one in (45) due to the symmetry of the canonical form (7).

From the matrix of coefficient one gets

$$|\det J| = \mathcal{C} \prod_{h < k=1}^{n_A} (\nu_h^2 - \nu_k^2)^2 \prod_{j=1}^{n_A} \nu_j^2 (\nu_j^2 - 1)^{(n_B - n_A)},$$

where  $\mathcal{C}$  is a constant factor. Finally, the invariant measure in Eq. (10) is obtained by inserting the differentials  $d\theta, d\nu_1, \dots, d\nu_{n_A}$ , and identifying the factors depending on the local degrees of freedom

$$\begin{aligned} d\mu(\alpha_A) &= d\xi_A \prod_{h < k=1}^{n_A} d^2 M_{Ahk} d^2 N_{Ahk} \prod_{i=1}^{n_A} d^2 M_{Aii}, \\ d\mu(\alpha_B) &= d\xi_B \prod_{h < k=1}^{n_B} d^2 M_{Bhk} d^2 N_{Bhk} \prod_{i=1}^{n_A} d^2 M_{Bii} dN_{Bii} \\ &\quad \times \prod_{j=1}^{n_A} \prod_{l=n_A+1}^{n_B} d^2 P_{B'jl} d^2 P_{B'jl} d^2 Q_{B'jl} d^2 Q_{B'jl} \prod_{p \leq q=n_A+1}^{n_B} d^2 M_{B'pq}. \end{aligned}$$

### B. Proof of Theorem 2

In order to derive the explicit expression of the Haar measure on the group of  $n$ -mode homogeneous Gaussian unitaries, we apply the unitary transformations on the vacuum state. Using the Euler decomposition in Eq. (11) we obtain the  $n$ -mode homogeneous Gaussian state

$$|\psi_G\rangle = \mathcal{U}_G |0\rangle = e^{-i\theta} \exp \left( -i \sum_{i,j=1}^n T_{ij} a_i^\dagger a_j \right) \exp \left( \sum_{k=1}^n s_k a_k^2 - s_k (a_k^\dagger)^2 \right) |0\rangle,$$

where we have used  $\exp\left(-i\sum_{i,j=1}^n T'_{ij}a_i^\dagger a_j\right)|0\rangle = |0\rangle$ . Then, by variation of the parameter  $\theta$ , of the elements of  $T$  and of the parameters  $s = (s_1, s_2, \dots, s_n)$ , we obtain a set of vector valued one-forms. By proceeding as in Section IV A, these one-forms can be used to derive an explicit expression for the invariant measure on the manifold of homogeneous Gaussian states. Then, the Haar measure on the group of homogeneous Gaussian unitaries can be readily obtained from the latter.

By variation of the parameter  $\theta$ , we obtain the vector-valued one-form

$$d\theta|0\rangle. \quad (56)$$

The variations of the parameters  $s$  yield

$$ids_k \sqrt{2}|k_2\rangle, \quad (57)$$

where  $|k_2\rangle = 2^{-1/2}(a_k^\dagger)^2|0\rangle$ . The variations of the parameters  $T_{kk}$  yield the one-forms

$$dT_{kk} \left[ 2^{-1/2} \sinh 2s_k |k_2\rangle + (\sinh s_k)^2 |0\rangle \right]. \quad (58)$$

Similarly, by variations of the parameters  $\text{Re}(T_{hk})$ , we obtain

$$d\text{Re}(T_{hk}) \sinh(s_h + s_k) |hk\rangle, \quad h < k \leq n, \quad (59)$$

and the variations of the parameters  $\text{Im}(T_{hk})$ , yield

$$d\text{Im}(T_{hk}) \sinh(s_k - s_h) |hk\rangle, \quad h < k \leq n, \quad (60)$$

where  $|hk\rangle = a_h^\dagger a_k^\dagger |0\rangle$ .

The volume form generated by these one-forms is by construction the invariant measure on the considered submanifold of Gaussian states. First, we notice that the volume forms generated by the variations of the matrix elements of  $T$  and  $T'$  are

$$\begin{aligned} \prod_{k=1}^n dT_{kk} \prod_{i<j=1}^n d^2 T_{ij} &= d\mu(U), \\ \prod_{k=1}^n dT'_{kk} \prod_{i<j=1}^n d^2 T'_{ij} &= d\mu(U'), \end{aligned}$$

where  $d\mu(U)$  and  $d\mu(U')$ , with  $U = \exp(-iT)$  and  $U' = \exp(-iT')$ , denote the Haar measure on the unitary group  $U(n)$ . Then, from the one-forms (56), (57), (58), (59), (60), we are able to derive the following expression for the Haar measure on the group of  $n$ -mode homogeneous Gaussian unitaries:

$$d\mu(\mathcal{U}_G) = \prod_{h<k=1}^n |\lambda_h - \lambda_k| \prod_{j=1}^n d\lambda_j d\mu(U) d\mu(U'),$$

with  $\lambda_k = \cosh 2s_k$ .

### C. Proof of Lemma 1

For a given  $n$ -mode homogeneous Gaussian state  $|\psi_G\rangle$ , and a bipartition of the system defined by two disjoint sets of canonical operators

$$\{a_{Ak}, a_{Ak}^\dagger\}_{k=1, \dots, n_A}, \quad \{a_{Bk}, a_{Bk}^\dagger\}_{k=1, \dots, n_B},$$

$n_A + n_B = n$ , we consider the mean value of the energy of one of the two subsystems. The non-homogeneous case can be analyzed in a similar way, and gives rise to an additional term in the mean energy. To fix the ideas we consider the mean energy of subsystem  $A$ ,

$$\mathcal{E}_A = \frac{1}{2} \sum_{k=1}^{n_A} \langle \psi_G | (a_{Ak}^\dagger a_{Ak} + a_{Ak} a_{Ak}^\dagger) | \psi_G \rangle,$$

and restrict to the case of balanced bipartition,  $n_A = n_B = n/2$ .

Using the canonical form (7) we get

$$\begin{aligned}\mathcal{E}_A &= \frac{1}{2} \sum_{k=1}^{n/2} \langle \psi_G^c | \mathcal{U}_G^{A\dagger} \left( a_{A_k}^\dagger a_{A_k} + a_{A_k} a_{A_k}^\dagger \right) \mathcal{U}_G^A | \psi_G^c \rangle \\ &= \frac{1}{2} \sum_{k=1}^{n/2} \langle \psi_G^c | \left( a'_{A_k}{}^\dagger a'_{A_k} + a'_{A_k} a'_{A_k}{}^\dagger \right) | \psi_G^c \rangle,\end{aligned}$$

where  $a'_{A_k} = \mathcal{U}_G^{A\dagger} a_{A_k} \mathcal{U}_G^A$  are  $a'_{A_k}{}^\dagger = \mathcal{U}_G^{A\dagger} a_{A_k}^\dagger \mathcal{U}_G^A$  are linear combinations of the operators  $\{a_{A_k}, a_{A_k}^\dagger\}_{k=1, \dots, m}$ . Their explicit form can be written starting from the Euler decomposition of the homogeneous Gaussian unitary, see Eq. (11),

$$\mathcal{U}_G^A = e^{-i\theta} \exp \left( -i \sum_{i,j=1}^{n/2} T_{Aij} a_{A_i}^\dagger a_{A_j} \right) \exp \left( \sum_{k=1}^{n/2} s_{A_k} a_{A_k}^2 - s_{A_k} (a_{A_k}^\dagger)^2 \right) \exp \left( -i \sum_{i,j=1}^{n/2} T_{A'ij} a_{A_i}^\dagger a_{A_j} \right).$$

Then, Eqs. (12), (13) imply

$$\mathcal{E}_A = \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{A h k}|^2 \lambda_{A h} \nu_k,$$

where  $U_A = \exp(-iT_A)$ ,  $\lambda_{A_k} = \cosh 2s_{A_k}$ , and  $\nu_k$ 's are the symplectic eigenvalues. The analogous expression is obtained for the mean energy of subsystem  $B$ ,

$$\mathcal{E}_B = \frac{1}{2} \sum_{h,k=1}^{n/2} |U_{B h k}|^2 \lambda_{B h} \nu_k.$$

The local mean energies take a particular simple form for the submanifold of states considered in Section III D. In that case, from Eqs. (19), (20) we get  $\lambda_{A_k} = \lambda_{B_k} = 1$ , which in turn implies

$$\mathcal{E}_A = \mathcal{E}_B = \frac{1}{2} \sum_{k=1}^{n/2} \nu_k.$$

#### D. Proof of Theorem 3

To derive the invariant measure in Eq. (21), we proceed along the same steps of Section IV A, with the difference that only the terms which are compatible with the form of the Hamiltonian in Eq. (18), and hence compatible with the local Gaussian unitaries in Eqs. (19), (20), have to be retained. Thus, the invariant measure is generated by the one-forms (26), (37)-(39), (44)-(46), (49)-(50), and (55). This yields the expression in Eq. (21) where the factors depending on the local degrees of freedom are explicitly given by

$$\begin{aligned}d\mu(\bar{\alpha}_A) &= \prod_{h < k=1}^{n_A} d^2 N_{A h k}, \\ d\mu(\bar{\alpha}_B) &= \prod_{h < k=1}^{n_B} d^2 N_{B h k} \prod_{i=1}^{n_A} dN_{B ii} \prod_{j=1}^{n_A} \prod_{l=n_A+1}^{n_B} d^2 Q_{B' j l} d^2 Q_{B' j l}.\end{aligned}$$



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